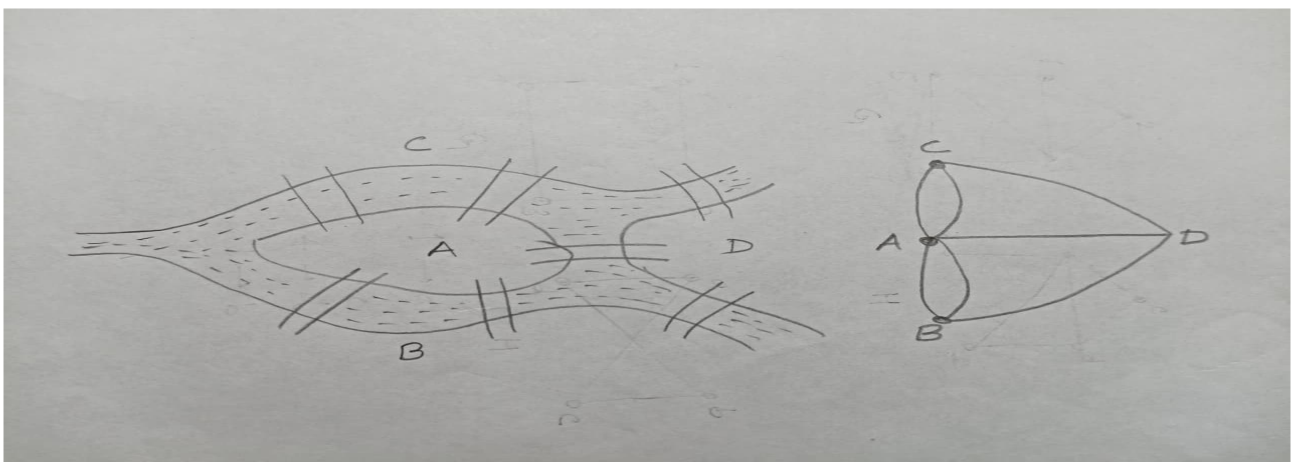
Lecture 15 (Eulerian and Hamiltonian Cycles)

1 Eulerian Graph

The Pregel River in Königsberg, Prussia (now known as Kaliningrad, part of Russia), divided the town into four distinct land regions, denoted by A, B, C, and D. These regions were connected by seven bridges: two bridges between A and C, two between A and B, one between A and D, one between B and D, and one between C and D, as illustrated in the figure.
A natural question arises: Is it possible to start at some location in the town, cross each bridge exactly once (without repeating any bridge), and return to the starting point?
This problem was solved in 1736 by the Swiss mathematician Leonhard Euler and is widely regarded as the origin of graph theory. Euler translated the problem into a multigraph, where land regions are represented as vertices and bridges as edges, as shown in the corresponding figure. He proved that such a traversal is possible if and only if every vertex of the graph has even degree. This fundamental result led to the concept of an Eulerian cycle in graph theory.



Definition 1. Let G be a graph. A trail in G is called an Eulerian trail if it contains all the edges of G. A graph G is said to be Eulerian if it has a closed Eulerian trail.

Remark. Note that a graph may contain an Euler trail but it need not be Eulerian. For example, consider the graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and $E(G) = \{13, 23, 34, 35, 45\}$. In this graph $1 - 3 - 4 - 5 - 3 - 2$ is an Eulerian trail but G has no Eulerian closed trail.

Theorem 1.1. Let G be a connected graph. Then the following statements are equivalent.

1. G is Eulerian.

2. Every vertex of G has even degree.

3. The set of edges can be partitioned into cycles.

Proof: $1 \Rightarrow 2$: Let G be Eulerian graph. Then G has an Eulerian closed trail T. Note that for each vertex v, the trail enters through an edge and departs from v through another edge. Thus, at each stage, the process of entering and exiting contributes 2 to the degree v. Since each edge appears exactly once, the degree of v is even.

$2 \Rightarrow 3$: Since G is connected and the degree of each vertex is even, the graph is not a tree. So there is at least one cycle C_1 in G. If C_1 is not G. Let G_1 be the subgraph (possibly disconnected) of G after deleting the edges in C_1. Since each vertex in a cycle has degree 2, the degree of each vertex in G_1 is even, and as before, it has a cycle C_2. Let $G_2 = G_1 - E(C_2)$. We repeat the process of identifying the cycles until we get the graph $G_k = G - E(C_1) - E(C_2) - \cdots - E(C_k)$ with no edges. Thus, the set of edges of these cycles gives the required partition.

$3 \Rightarrow 1$: Suppose the set of edges in a connected graph G is the disjoint union of k cycles. Consider any one of these cycles, say cycle C_1. Since the graph is connected, there is a cycle, say C_2, such that the two cycles have a vertex v_1 in common. Let Q_{12} be the circuit that consists of all the edges in these two cycles. As before, there is a cycle C_3 such that this cycle and the circuit Q_{12} have no edge common but do have vertex v_2 in common. Let Q_{123} be the circuit that contains all the edges of these three edge-disjoint cycles. We repeat this process until we get a circuit that contains all the edges of the graph. This graph is Eulerian.

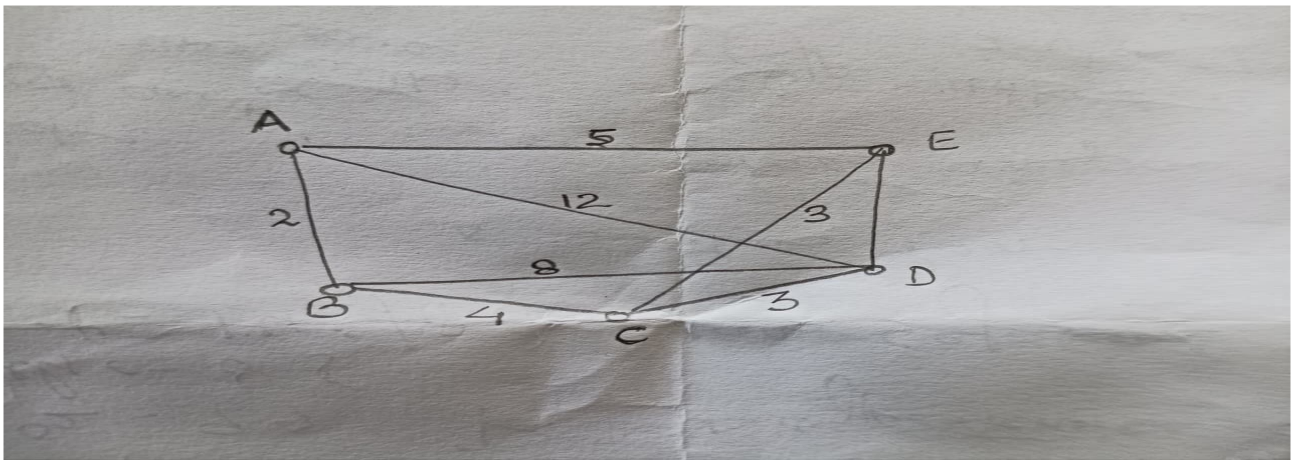
We can obtain an Eulerian circuit in an Euler graph G as the set of edges is partitioned into cycles by the following algorithm:

Step 1: Start from any vertex v and construct a cycle C.
Step 2: If C contains all the edges of G, stop. Otherwise choose a vertex w common to C and the subgraph $G' = G \setminus C$.
Step 3: Starting from w construct a cycle C' in G'.
Step 4: Combine the edges of C and C' to obtain a new circuit in G. This new circuit is C. Now return to Step 2.

2 Hamiltonian Graph

Consider the traveling salesman problem (TSP). Suppose a salesman must visit the cities A, B, C, D, and E starting from a certain one (e.g., the hometown) and returning to the same city. The challenge is to minimize the total trip length.

Let $d(X, Y)$ denote the distance of a road between cities X and Y. Suppose $d(A, B) = 2$, $d(A, D) = 12$, $d(A, E) = 5$, $d(B, C) = 4$, $d(B, D) = 8$, $d(C, D) = 3$, $d(C, E) = 3$, and $d(D, E) = 10$. Let us deal with this problem through a graph. Consider the cities as vertices and roads as edges. Then we have the following graph.



Clearly the goal can be achieved if such cyclic trip exists. Further, if there is more than one such trip, we need to choose the one with the minimum length.
For example, the cyclic trip $P_1 := (A, B, C, D, E, A)$ has total length 24 while the trip $P_2 := (A, B, C, E, D, A)$ has total length 31. Thus, the salesman will prefer the trip P_1. These cyclic trips are obtained from the concept of a Hamiltonian cycle.

Definition 2. Let G be a graph. A cycle in G is said to be Hamiltonian if it contains all vertices of G. If G has a Hamiltonian cycle, then G is called a Hamiltonian graph.

Example 1. *1. For each positive integer $n \geq 3$, the cycle C_n is Hamiltonian.*

2. For each positive integer $n \geq 3$, the cycle K_n is Hamiltonian.

3. The graphs corresponding to all platonic solids are Hamiltonian.

Proposition 1. *The Petersen graph is not Hamiltonian.*

Proof: Suppose G is Hamiltonian. So, G contains cycle $C_{10} = (1, 2, 3, \ldots, 10, 1)$ as a subgraph. Note that degree of each vertex 3 and $g(G) = 5$. Consider the vertices 1,2 and 3.

In the view of $g(G)$, the vertex 1 can be adjacent to only one of the vertices 5,6, or 7. If 1 is adjacent to 5, then the possible third vertex that is adjacent to 10 will create cycles of length 3 or 4. Similarly, if 1 is adjacent to 7, then there is no choice for the possible third vertex that can be adjacent to 2. So, let 1 be adjacent to 6. Then, 2 must be adjacent to 8. In this case, note that there is no choice for the third vertex that can be adjacent to 3. Thus, the Petersen graph is non-Hamiltonian.

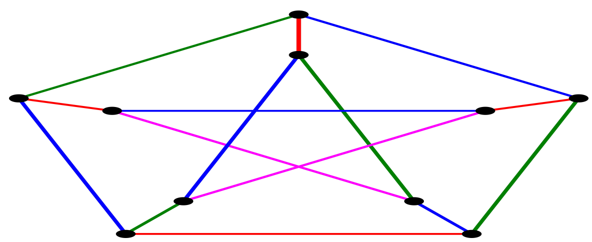


Figure 1: Petersen graph

Theorem 2.1. (Ore's Theorem 1960: A sufficient condition for a graph to be Hamiltonian). Let G be a simple graph with $|G| \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u and v. Then G is Hamiltonian.

Proof: Suppose G is not Hamiltonian. Add edges to G until we get a graph G' such that G' is maximal non-Hamiltonian, i.e., any missing edge makes it Hamiltonian. Let u and v be non-adjacent vertices in G'. Then by hypothesis, $\deg(u) + \deg(v) \geq n$.
Now consider the graph $G'' = G' \cup \{uv\}$. Then by maximality of G', G'' is Hamiltonian and hence has a Hamiltonian cycle containing the edge uv. Thus, removing the edge uv in the cycle gives a Hamiltonian path in G' from u to v. Let it be $P := u = v_1 - v_2 - \cdots - v_n = v$.
Since $\deg(u) + \deg(v) \geq n$, there must exist indices i such that $u \sim v_i$ and $v \sim v_{i-1}$ for $i \neq \{1, n\}$. This gives a Hamiltonian cycle $C := u = v_1 - v_2 - \cdot - v_{i-1} - v - v_{n-1} - v_{n-2} - \cdots - v_i - v_1 = u$ in G'. A contradiction. So G is Hamiltonian.

Theorem 2.2. (Dirac's Theorem 1952: A sufficient condition for a graph to be Hamiltonian). Let G be a simple graph with $|G| \geq 3$ such that $\deg(v) \geq \frac{n}{2}$. Then G is Hamiltonian.

Proof. Since $\deg(v) \geq \frac{n}{2}$ for all the vertices v in G, for every pair of non-adjacent vertices

$\{x, y\}$, $\deg x + \deg y \geq \frac{n}{2} + \frac{n}{2} = n$. So, by Ore's theorem, G is Hamiltonian.

Remark. The converses of the above theorems are false. The cyclic graph with five or more vertices is Hamiltonian, but the degree of every vertex in that graph is only 2. Also, the sum of the degrees of every pair of vertices is 4.

Example. Let $G = (V, E)$ be a graph with $|E| \geq 3$. Show that if $|E| = \frac{(n-1)(n-2)}{2} + 2$, then G is Hamiltonian. Is the converse true?

Solution. If G is a complete graph, it is Hamiltonian. Otherwise, let G' be a maximal non-Hamiltonian graph (obtained from G by adding edges in G such that adding one more edge makes G' Hamiltonian). Let u, v be two non-adjacent vertices in G'. Then $\deg(u) + \deg(v) \leq (n-1)$, otherwise by Ore's theorem the graph would be Hamiltonian. Then

$$|E(G')| \leq \binom{n-2}{2} + (n-1) = \frac{(n-1)(n-2)}{2} + 1.$$

Note that

$$|E(G')| \geq |E(G)| = \frac{(n-1)(n-2)}{2} + 2.$$

A contradiction.

However, converse need not be true. A graph can be Hamiltonian with far fewer edges. Let C_5.

References:

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2: K. H. Rosen; Discrete mathematics and its applications, Tata McGraw-Hill.

3: V. K. Balakrishnan; Graph Theory (Schaum's outlines) Discrete mathematics and its applications, McGraw-Hill.